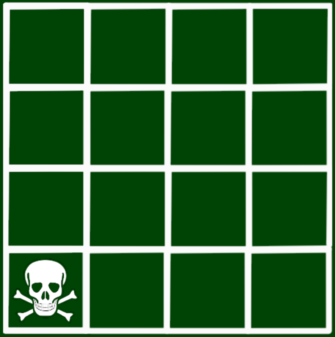
CS 2

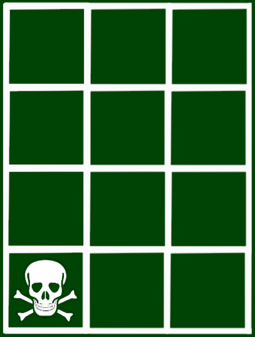
Introduction to Programming Methods

Games: Part I

Chomp!

Two players. On their turn, a player chooses a square on the chocolate bar and eats all squares above and to the right. The player who is forced to eat the "poison" square in the bottom left loses.





If we assume optimal play, who wins in each game? Why?

Why Mathematical Games?

- AI testbed

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- Beautiful Mathematical theory

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- Fusion of math and programming

Games in the News: 1997

Garry Kasparov, left, playing against the I.B.M. Deep Blue computer in the sixth and final game of a match in New York in May 1997. The computer's pieces were moved by Joseph Hoane, right, an I.B.M. scientist. Stan Honda/Agence France-Presse — Getty Images

Computer Checkers Program Is Invincible

By Kenneth Chang

July 19, 2007

For an exercise in futility, go play checkers against a computer program named Chinook.

Developed by computer scientists at the University of Alberta in Canada, Chinook vanquished human competitors at tournaments more than a decade ago. Now, in an article published today on the Web site of the journal Science, the scientists report that they have rigorously proved that Chinook, in a slightly improved version, cannot ever lose. An opponent, no matter how skilled, practiced or determined, can at best achieve a draw.

Daily Report: AlphaGo Shows How Far Artificial Intelligence Has Come

The Google artificial intelligence program AlphaGo beat the top-ranked Chinese Go player, Ke Jie, in Wuzhen, China, in the first game of a three-game match. Wu Hong/European Pressphoto Agency

NEWS · 30 OCTOBER 2019

Google AI beats top human players at strategy game *StarCraft II*

DeepMind's AlphaStar beat all but the very best humans at the fast-paced sci-fi video game.

NEWS · 30 NOVEMBER 2020

‘It will change everything’: DeepMind’s AI makes gigantic leap in solving protein structures

Google’s deep-learning program for determining the 3D shapes of proteins stands to transform biology, say scientists.

Definition (Combinatorial Game)

A two-player game with **perfect information** (i.e., no hidden state) and no **randomness** in which the players alternate turns. Certain positions in the game are denoted “terminal” and the game ends if any of these positions are reached.

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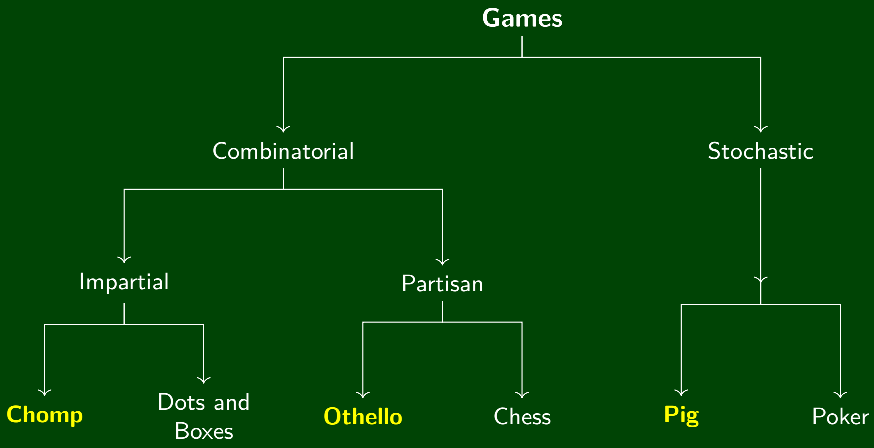
Definition (Partisan Game)

A combinatorial game in which players may have different moves available based on their identity.


Games
Combinatorial
Stochastic
Impartial
Partisan
Chomp
Dots and Boxes
Othello
Chess
Pig
Poker

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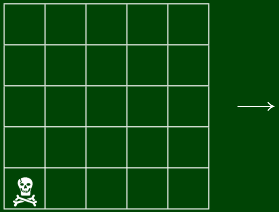
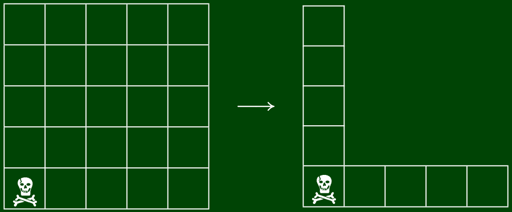
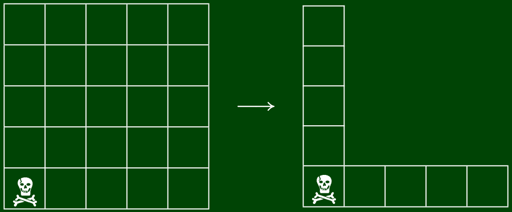
This series of topics very nicely underlines the core material in this course.

Chomp: Take 1







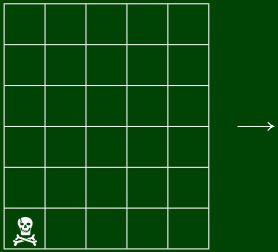
Mirroring

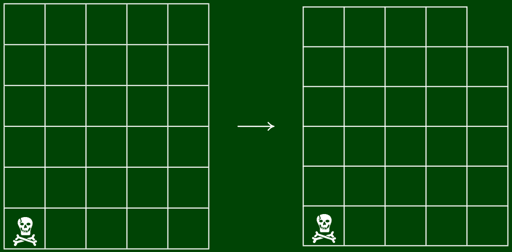
On a square Chomp! board, the first player has a winning strategy:

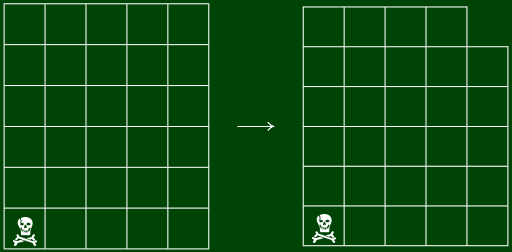
- Choose $(2,2)$ as the first move.
- For all remaining moves, **mirror** the other player across the diagonal.

Mirroring is highly effective!

Chomp: Take 2







Strategy Stealing

Claim: Player 1 has a winning strategy on a rectangular board.
Proof: Consider the move that eats only the top right square. If this positions "leads" to a win, make that move. If it doesn't, then player 2 has a winning move. Note that since the remaining board is a strict sub-board of the original one, the first player could have made that winning move!

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Standard Terminology

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Let's play a game...

Nim

Two player game. On your turn, you may take any number of coins **from a single pile**. If you cannot take at least one coin, you lose.

One Pile Nim

Two player game. On your turn, you may take any number of coins **from the pile**. If you cannot take at least one coin, you lose.

Too Hard? How about one pile?

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Too Easy...

- $\mathcal{P} = \{0\}$
- $\mathcal{N} = \mathbb{N} \setminus \{0\}$

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How about three piles? How about n piles?

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      Let $X = x_0 \oplus \cdots \oplus x_n = (d_0 d_1 \cdots d_q)_2$. We aim to find a position in which a single pile is (1) smaller and (2) makes $X = 0$.
        - Consider the left-most bit of X, where $d_i = 1$. (This is exists because $X \neq 0$.)
        - Choose x_j such that the ith bit of x_j is also 1. (If they were all zeroes, d_i would also be zero.)
        - Choose x_j' such that the ith bit of x_j is flipped iff $d_j = 1$.

      Trivially, $X \oplus x_j \oplus x_j' = 0$, because we rigged it that way above. Also, $x_j' < x_j$, because the most significant bit we flipped was $1 \to 0$.