CS 2

Introduction to Programming Methods

Asymptotic Analysis


```
public void run() {
   //for (int i = 0; i < 1000000; i++) {
       //doLongCalculation();
       //anotherAnalysis();
       //solvePNP();
   //}
    System.out.Println("Done!");
}
```

Outline

Comparing Programs

What does it mean to have an "efficient program"?

```java
System.out.println("hello");
```

vs.

```java
System.out.print("h");
System.out.print("e");
System.out.print("l");
System.out.print("l");
System.out.println("o");
```

OUTPUT

```
>> left average run time is 1000 ns.
>> right average run time is 5000 ns.
```

We're measuring in NANOSECONDS!

Both of these run **very very** quickly. The first is definitely better style, but it's not "more efficient."

`hasDuplicate`

Given a **sorted int array**, determine if the array has a duplicate.

Algorithm 1

hasDuplicate

Given a **sorted int array**, determine if the array has a duplicate.

Algorithm 1

For each **pair of elements**, check if they're the same.

Algorithm 2

hasDuplicate

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Algorithm 2

For each **element**, check if it's equal to the one after it.

Why Not Time Programs?

Comparing Programs: Timing

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Algorithm 1

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Algorithm 2

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Why Not Time Programs?

Timing programs is prone to error (not **reliable** or **portable**):

- Hardware: processor(s), memory, etc.
- OS, Java version, libraries, drivers
- Other programs running
- Implementation dependent
- Can we even time an algorithm?

hasDuplicate

Given a **sorted int array**, determine if the array has a duplicate.

Example

```java
public int stepsHasDuplicate1(int[] array) {
   int steps = 0;
   for (int i=0; i < array.length; i++) {
      for (int j=0; j < array.length; j++) {
         steps++; // The if statement is a step
         if (i != j && array[i] == array[j]) {
            return steps;
         }
      }
   }
   return steps;
}
```

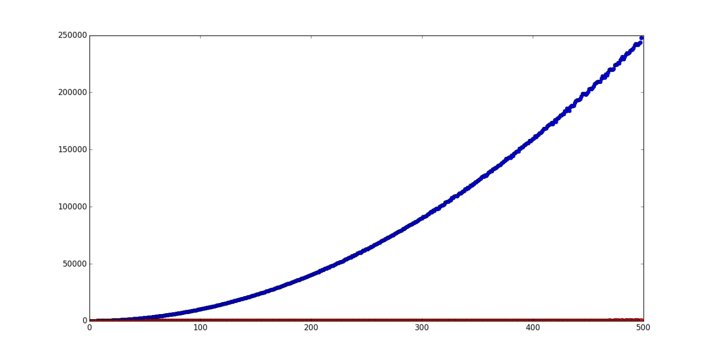
OUTPUT

```
>> hasDuplicate1 average number of steps is 9758172 steps.
>> hasDuplicate2 average number of steps is 170 steps.
```

Why Not Count Steps in Programs?

- Can we even count steps for an algorithm?
- We must do this via **testing**; so, we may miss worst-case input!
- We must do this via **testing**; so, we may miss best-case input!

Instead, let's try running on arrays of size 1, 2, 3, . . . , 1000000, and plot:



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We want to compare **algorithms**, not programs. In general, there are many answers (clarity, security, etc.). Performance (space, time, etc.) are generally among the most important.

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- Only consider large inputs (any algorithm will work on 10)
- Answer will be independent of CPU speed, programming language, coding tricks, etc.
- Answer is general and rigorous, complementary to "coding it up and counting steps on some test cases"
- Can do analysis before coding!

Comparing Code: Analytically

Basic Operations take "some amount of" Constant Time

- Arithmetic (fixed-width)
- Variable Assignment
- Access one Java field or array index
- etc.

(This is an approximation of reality: a very useful "lie".)

Complex Operations

Consecutive Statements. Sum of time of each statement
Conditionals. Time of condition + `max(ifBranch, elseBranch)`
Loops. Number of iterations $*$ Time for Loop Body
Function Calls. Time of function's body
Recursive Function Calls. Solve Recurrence

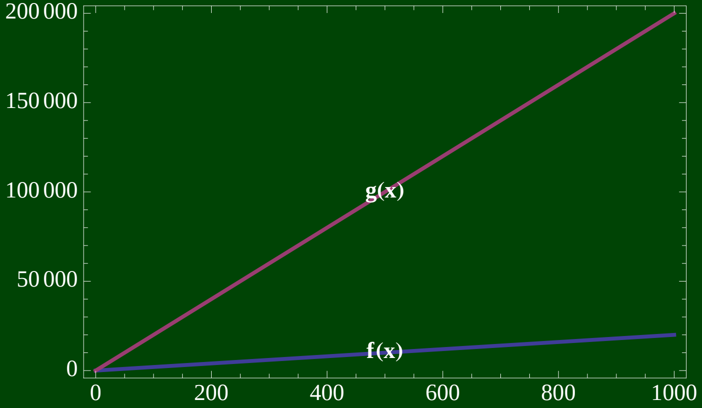
```
public boolean hasDuplicate1(int[] array) {
   for (int i=0; i < array.length; i++) { // 1
      for (int j=0; j < array.length; j++) { // 1
         if (i != j && array[i] == array[j]) { // 1
            return true;             // 1
         }
      }
   }
   return false;                     // 1
}
```

$\} 2 \quad \} 2N+1 \quad \} (2N+1)(N+1)$

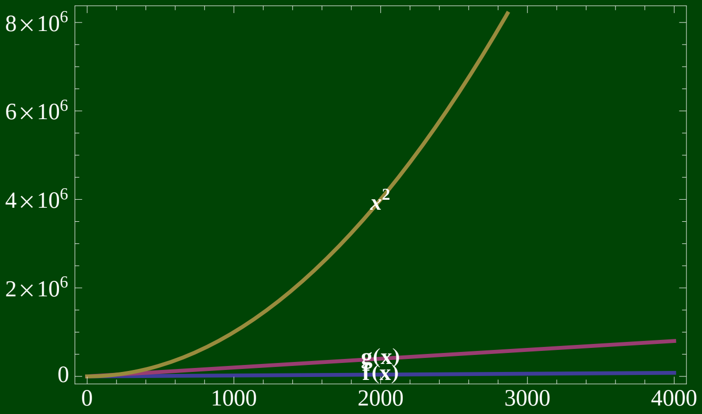
```
public boolean hasDuplicate2(int[] array) {
   for (int i=0; i < array.length – 1; i++) { // 1
      if (array[i] == array[i+1]) { // 1
         return true;            // 1
      }
   }
   return false;                 // 1
}
```

$\} 2 \quad \} 2N \quad \} 2N+1$

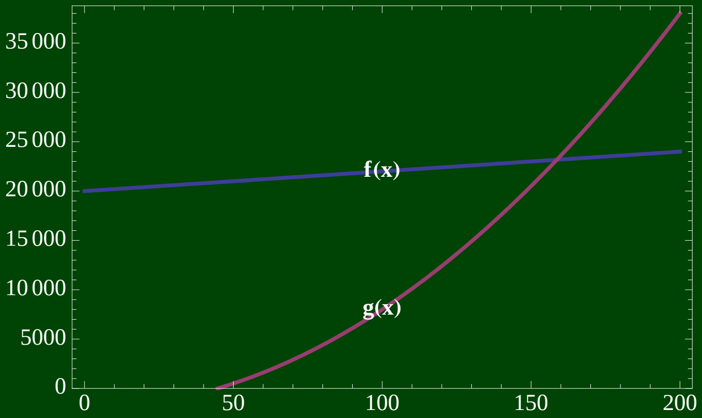
Outline



Should we consider these “the same”?



Probably a good idea, since they seem to be growing at the same rate. For reference, the function that dwarfs them both is x^2.



Here's two functions, $f(x)$ and $g(x)$. Ultimately, $g(x)$ will grow much faster than $f(x)$, but at the beginning, it is smaller.

We'd like to be able to compare two functions. Intuitively, we want an operation like "$\leq$" (e.g. $4 \leq 5$), but for functions.

If we have f and $4f$, we should consider them the same:

$f \leq g$ when...

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If we have f and $4f$, we should consider them the same:

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$f \leq cg$ where c is a constant and $c \neq 0$.

We also care about **all values of the function** that are **big enough**:

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Big-Oh Examples

True or False?

(1) $4 + 3n \in \mathcal{O}(n)$

(2) $4 + 3n = \mathcal{O}(1)$

(3) $4 + 3n$ is $\mathcal{O}(n^2)$

(4) $n + 2\log n \in \mathcal{O}(\log n)$

(5) $\log n \in \mathcal{O}(n + 2\log n)$

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Big-Oh Gotchas

- $\mathcal{O}(f)$ is a **set**! This means we should treat it as such.
- If we know $f(n) \in \mathcal{O}(n)$, then it is also the case that $f(n) \in \mathcal{O}(n^2)$, and $f(n) \in \mathcal{O}(n^3)$, etc.
- Remember that small cases, really don't matter. As long as it's **eventually** an upper bound, it fits the definition.

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Okay, but we haven't actually shown anything. Let's **prove** (1) and (2).

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We say a function $f : A \to B$ **is dominated by** a function $g : A \to B$ when:

$$\exists(c, n_0 > 0).\ \forall(n \geq n_0).\ f(n) \leq cg(n)$$

Formally, we write this as $f \in \mathcal{O}(g)$.

We want to prove $4 + 3n \in \mathcal{O}(n)$. That is, we want to prove:

$$\exists(c, n_0 > 0).\ \forall(n \geq n_0).\ 4 + 3n \leq cn$$

Proof Strategy

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- Choose a c, n_0 that work.

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Proof

Big-Oh Proofs

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Proof Strategy

- Choose a c, n_0 that work.
- Prove that they work for all $n \geq n_0$.

Proof

Choose $c = 5$ and $n_0 = 5$. Then, note that $4 + 3n \leq 4n \leq 5n$, because $n \geq 5$. It follows that $4 + 3n \in \mathcal{O}(n)$.

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Scratch Work

We want to choose a c and n_0 such that $4 + 3n + 4n^2 \leq cn^3$. So, manipulate the equation:

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We want to choose a c and n_0 such that $4 + 3n + 4n^2 \leq cn^3$. So, manipulate the equation:

$$4 + 3n + 4n^2 \leq 4n^3 + 3n^3 + 4n^3 = 11n^3$$

For this to work, we need $4 \leq 4n^3$ and $3n \leq 3n^3$. $n \geq 1$ satisfies this.

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Proof

Choose $c = 11$ and $n_0 = 1$. Then, note that $4 + 3n + 4n^2 \leq 4n^3 + 3n^3 + 4n^3 = 11n^3$, because $n \geq 1$. It follows that $4 + 3n + 4n^2 \in \mathcal{O}(n^3)$.

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Definition (Big-Omega)

We say a function $f : A \to B$ **dominates** a function $g : A \to B$ when:

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Formally we write this as $f \in \Omega(g)$.

Definition (Big-Theta)

We say a function $f : A \to B$ **grows at the same rate as** a function $g : A \to B$ when: $f \in \mathcal{O}(g)$ and $f \in \Omega(g)$
Formally we write this as $f \in \Theta(g)$.

Important: You need not use the same c value for $\mathcal{O}$ and Ω to prove Θ.

True or False?

(1) $4+3n \in \Theta(n)$

(2) $4+3n$ is $\Theta(n^2)$

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If you want to say "f is a tight bound for g", **do not use** $\mathcal{O}$–use Θ.

Remember, we're analyzing the **worst** case **time**! What else can we analyze?

- Space?
- Average Case?
- Best Case?
- Time **over multiple operations**?

Because $\log_2$ is so common in CS, we abbreviate it $\lg$. When it comes to Big-Oh, **all log bases are the same**:

Recall the $\log$ change of base formula:

$$\log_b(x) = \frac{\log_d(x)}{\log_d(b)}$$

Then, to show $\log_b(n) \in \mathcal{O}(\log_d(n))$, note the following:

For all $n \geq 0$, we have $\log_b(x) = \frac{1}{\log_d(b)} \log_d(x)$.

Which is Better?

$n^{1/10}$ or $\log n$

- $\log n$ grows more slowly (Big-Oh)
- ...But the cross-over point is around 5×10^{17}

Today's Takeaways!

- There are many ways to compare algorithms
- Understand formal Big-Oh, Big-Omega, Big-Theta
- Be able to prove any of these