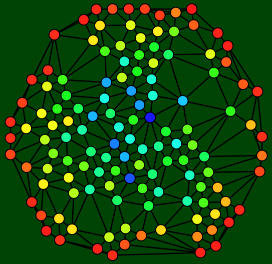
CS 2

Introduction to Programming Methods

Case Study: Union Find



A **disjoint sets** data structure keeps track of multiple sets which do not share any elements. Here's the ADT:

UnionFind ADT

`find(x)`	Returns a number representing the set that **x** is in.
`union(x, y)`	Updates the sets so whatever sets **x** and **y** were in are now considered the same sets.

Example

```
list = [1, 2, 3, 4, 5, 6];
UF uf = new UF(list); // State: {1}, {2}, {3}, {4}, {5}, {6}
uf.find(1);           // Returns 1
uf.find(2);           // Returns 2
uf.union(1, 2);       // State: {1, 2}, {3}, {4}, {5}, {6}
uf.find(1);           // Returns 1
uf.find(2);           // Returns 1
uf.union(3, 5);       // State: {1, 2}, {3, 5}, {4}, {6}
uf.union(1, 3);       // State: {1, 2, 3, 5}, {4}, {6}
uf.find(3);           // Returns 1
uf.find(6);           // Returns 6
```

This lecture is an excuse to walk through the genesis of a data structure. We know what we want to get, and we'd like to **create (and analyze) an efficient data structure** that meets our requirements.

To define a `UnionFind` data structure, we need three things:

- The idea of the data structure
- An implementation of `union`
- An implementation of `find`

For the duration of the lecture, we will assume we can identify each item with a number from 1 to n.

Let's start out easy. . .

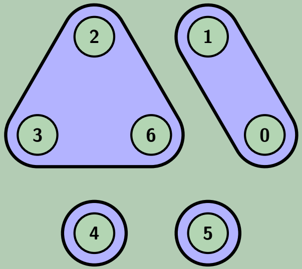
Implementation 1: A List of LinkedLists

Data Structure

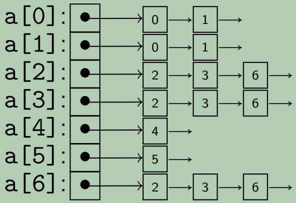
Type: `List<LinkedList<Integer>>`

Idea: A mapping from **id** → a list of **id**s in the same set

Pictorial View



Data Structure



find(x)

```
find(x) {
   return a[x].front;
}
```

union(x, y)

```
union(x, y) {
   ...
}
```

Implementation 1.5: A List of LinkedLists

Data Structure

Type: `List<LinkedList<Integer>>`

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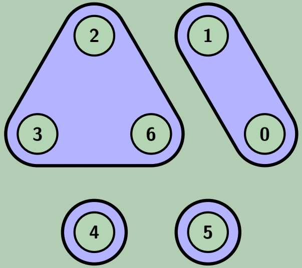
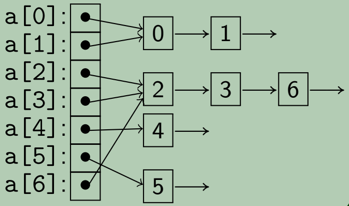
Pictorial View



Data Structure



find(x)

```
find(x) {
   return a[x].front;
}
```

union(x, y)

```
union(x, y) {
   curr = a[x].head;
   a[y].tail.next = curr;
   while (curr != null && curr.next != null) {
      a[curr.data] = a[y].head
      curr = curr.next;
   }
}
```

Implementation 1.5: A List of LinkedLists

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}
```

Asymptotic Analysis

- find(x) is $\mathcal{O}(1)$
- union(x, y) is $\mathcal{O}(\texttt{a[x].length})$

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}
```

Amortized Analysis

Consider any m find/union operations. The **worst** case is going to be that all the operations are all unions, but which unions?

Always union(LARGEST, y), because we have to traverse the first one.

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This ends up being $1+2+\cdots+n-1 = \frac{(n-1)n}{2}$.

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   }
}
```

Amortized Analysis

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$$\frac{(n-1)n}{2(n-1)} = \frac{n}{2}$$

This is horrible! How could this be better?

Data Structure

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OLD `union(x, y)`

```
union(x, y) {
   curr = a[x].head;
   a[y].tail.next = curr;
   while (curr != null && curr.next != null) {
      a[curr.data] = a[y].head;
      curr = curr.next;
   }
}
```

NEW `union(x, y)`

```
union(x, y) {
   if (a[x].length > a[y].length) {
      x, y = swap(x, y)
   }

   curr = a[x].head;
   a[y].tail.next = curr;
   while (curr != null && curr.next != null) {
      a[curr.data] = a[y].head
      curr = curr.next;
   }
}
```

Asymptotic Analysis

- `find(x)` is $\mathcal{O}(1)$
- `union(x, y)` is $\mathcal{O}(\min(\texttt{a[x].length}, \texttt{a[y].length}))$

Data Structure

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Amortized Analysis

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> Keep the sets as balanced as possible. This will get us the largest gurantee possible, as quickly as possible

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n sets of $1 \xrightarrow{\text{union}} n/2$ sets of $2 \xrightarrow{\text{union}} \ldots\ n/2^i$ sets of $2^i \xrightarrow{\text{union}} \ldots$

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So, now we compute:

$$\sum_{i=0}^{\lg n} \left(\frac{n}{2^{i+1}}\right) \text{cost}(\texttt{union}\left(2^i, 2^i\right))$$

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$$\sum_{i=0}^{\lg n}\left(\frac{n}{2^{i+1}}\right)\text{cost}(\texttt{union}\left(2^i,2^i\right)) = \sum_{i=0}^{\lg n}\left(\frac{n}{2^{i+1}}\right)2^i$$

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Since we remove one "gap" each time we union, there are $n-1$ union operations total.

So, the amortized cost of a union is:

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Since we remove one "gap" each time we union, there are $n-1$ union operations total.

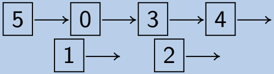
So, the amortized cost of a union is: $\dfrac{\frac{n}{2}\lg(n)}{n-1} \approx \lg(n)$.

Implementation 3: IMPLICIT Lists Unioned-By-Size

We started with a **list of linked lists**. Then, we realized that we could use **references to the same linked list** to save memory.

We can do even better. The idea is to use an “implicit list”.

Example (Explicit List)



Example (Implicit List)

3	-1	-1	4	-1	0
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]

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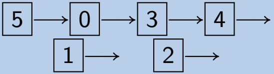

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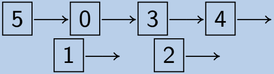

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Using An Implicit List

We need to store:

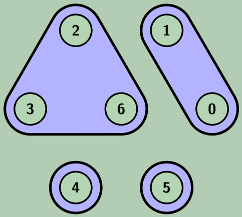
- pointers to get to the canonical member
- the size of the set

Data Structure

Type: An array

Idea: Each index has either the value of the "next" thing in its set or a negative number representing the size of the set

Pictorial View



Data Structure

-2	0	6	2	-1	-1	-3
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]

- "Non-canonicals" store "pointers"
- "Canonicals" store `-size`

Implementation

```
init(x) { a[x] = −1 }
```

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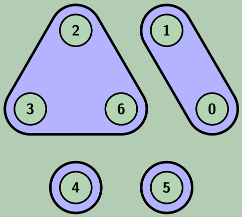

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Implementation

```
init(x) { a[x] = −1 }
find(x) {
   while(a[x] >= 0) {
      x = a[x]
   }
   return x
}
```

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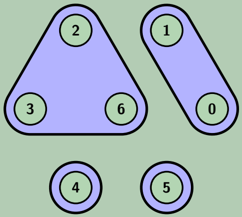

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   }
   return x
}

size(x) { return −a[find(x)] }
```

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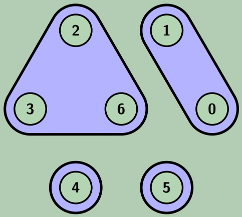

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}

size(x) { return −a[find(x)] }

union(x, y) {
   if (size(x) > size(y)) {
      x, y = swap(x, y)
   }
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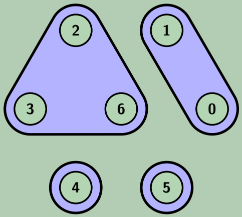

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union(x, y) {
   if (size(x) > size(y)) {
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   }

   // Now, we have: size(x) <= size(y)
   a[find(x)] = find(y)
```

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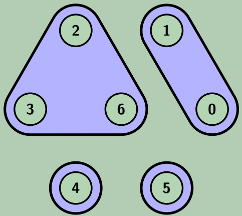

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   }

   // Now, we have: size(x) <= size(y)
   a[find(x)] = find(y)

   // Update the size
   a[find(y)] = size(x) + size(y)
}
```

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   a[find(x)] = find(y)

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```

- Assume we only call each size/find once.

Analyzing Implementation 3

-2	0	6	2	-1	-1	-3
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Implementation

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   // Now, we have: size(x) <= size(y)
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   // Update the size
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- Assume we only call each size/find once.
- Then, $\texttt{union(x, y)} \in \mathcal{O}(\texttt{find(x)} + \texttt{find(y)})$.

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- We claim that `find(x)` $\in \mathcal{O}(\lg n)$.
- To prove this, we will show the **height** of the tree resulting from some number of unions is $\mathcal{O}(\lg n)$
- (Sound familiar?)

Analyzing Implementation 3

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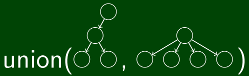

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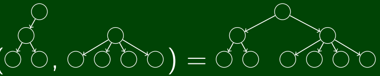
union(

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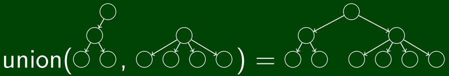

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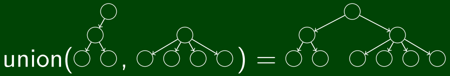

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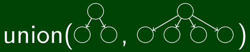

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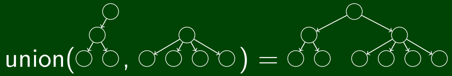

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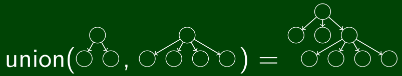

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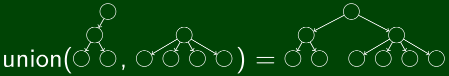

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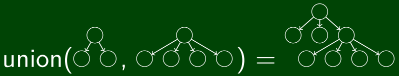

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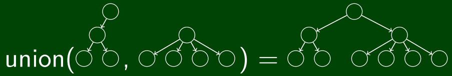

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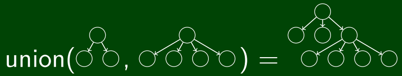

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  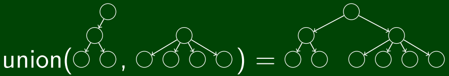

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  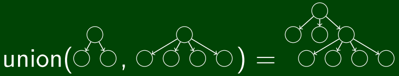

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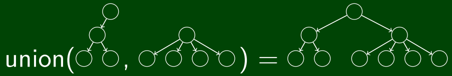

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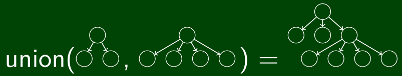

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  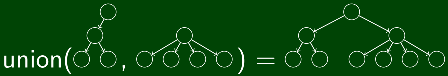

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  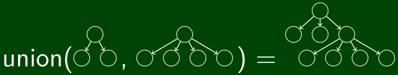

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(Notice this is nearly identical to the proof for AVL Trees, except with a slightly different recurrence.)

OLD `find(x)`

```
find(x) {
   while(a[x] >= 0) {
      x = a[x]
   }
   return x
}
```

NEW `find(x)`

```
find(x) {
   if (a[x] < 0) {
      return x
   }
   a[x] = find(a[x])
   return a[x]
}
```

In Words: Once we've **found** a node. . . save it.

Amortized Analysis of m `find` Operations?

Consider what we know:

- We know the worst case height of a tree is $\lg(n)$.
- We know it's difficult to make a tree of large height.
- We know that as soon as we access a path in a tree, it flattens the whole path

This **feels** like it should be better than $\lg(n)$, and it is.

We can use facts to show this, but its outside the scope of this lecture. Instead, we'll just talk about two bounds.

Upper Bound 1: `find`(x) is amortized $\mathcal{O}(\lg^*(n))$

Let 2STACK$(n) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\} n$

- 2STACK$(0) = 1$
- 2STACK$(1) = 2$
- 2STACK$(2) = 2^2 = 4$
- 2STACK$(3) = 2^{\left(2^2\right)} = 16$
- 2STACK$(4) = 2^{\left(2^{\left(2^2\right)}\right)} = 2^{16} = 65536$
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$2^{65536} =$ 20035299304068464649790723515602557504478254755697514l9...

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2650169737108940595563114530895061308809333481010382343429072
6318182294938211881266886950636476154702916504187191635158796
6347219442930927982084309104855990570159318959639524863372367
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This number has 19729 digits...

$\lg^*(n)$ is the **inverse function** of 2STACK.

1011613712423761426722541732055959202782129325725947146417224
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77506072339445587895905719156736

This number has 19729 digits. . .

$\lg^*(n)$ is the **inverse function** of 2STACK.

So, basically:

$$\lg^*(n) \le 5$$

But it gets better...

Upper Bound 2: `find`(x) is amortized $\mathcal{O}(\alpha(n))$

But it gets better...

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The Ackermann function grows even more quickly than 2STACK.

Inverse Ackermann function

But it gets better. . .

Upper Bound 2: `find`(x) is amortized $\mathcal{O}(\alpha(n))$

The Ackermann function grows even more quickly than 2STACK.

It turns out $\alpha(n)$, the **inverse Ackermann function** is also an upper bound. . .

But it gets better. . .

Upper Bound 2: `find`(x) is amortized $\mathcal{O}(\alpha(n))$

The Ackermann function grows even more quickly than 2STACK.

It turns out $\alpha(n)$, the **inverse Ackermann function** is also an upper bound. . .

Interestingly, **it is also a lower bound** for the disjoint data structures problem! We can't do better than the algorithm we came up with! (Just like with sorting!)